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# Effect of radiative heat transfer on the propagation of cylindrical shock waves

BY T. D. VARMA*

*Defence Science Laboratory, Delhi, India*

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A similarity solution for the propagation of intense cylindrical shock waves has been obtained. By the introduction of radiative heat flux the singularity in a solution of Lin (1954) is removed. It is observed that the type of configuration obtained and also the rate of propagation of the bounding shock depends on the amount of energy released and the density of the ambient air.

## INTRODUCTION

Under certain physical conditions it is some times found that temperatures are high enough for the hydrodynamics to be affected by radiation terms namely radiation flux, energy and pressure. Equations governing the flow in these circumstances were formulated by Thomas (1930). Marshak (1958) obtained similarity solutions of the radiation hydrodynamic equations for particular cases, when there was a plane symmetry and radiation energy and pressure were negligible although the flux was important, the cases he considered are those of (a) constant density (b) constant pressure and (c) power law time dependence of temperature. The extent to which problems in radiation hydrodynamics might be tackled by similarity methods was investigated by Elliot (1960). In particular he studied the problem of an intense point explosion and it was found that the type of configuration obtained as well as the rate of propogation of the bounding shock was dependent on the amount of energy released, and on the density of the ambient air ; in this respect the solution differed from that obtained using only the hydrodynamic equations.

Lin(1954) and Sedov (1959) investigated the flow behind a cylindrical disturbance bounded by a strong shock wave which resulted from the instantaneous release of a finite amount of energy per unit length along a line. An exact analytical solution of this problem was obtained by Chakraborty (1962). In these solutions there was a singularity at the axis of symmetry where both temperature and temperature gradient became infinite. This singularity could be removed by introducing heat flux in the problem.

In the present paper we have extended Elliot's analysis to investigate the flow behind a cylindrical shock wave. Radiative heat flux is included in the

*Present address : Defence Research Laboratory (Materials), Post Box No. 320, Kanpur-4.

problem, though we assume that the temperatures are low enough to justify neglecting radiation energy and pressure compared to material energy and pressure. We have used the diffusion approximation for the radiation flux. The losses of energy from the disturbance are neglected, which makes the total energy of the blast a constant

## Fundamental Equations

The flow is governed by the usual hydrodynamic equations with some modifications to include radiation flux. The radiation pressure and energy are neglected as compared to material pressure and energy, but at the same time it is assumed that the temperature of the matter or more precisely the radiation density is so high that the energy transfer is accomplished basically by radiation

The equations of continuity, momentum and energy for cylindrical symmetry are

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\rho u)=0 \qquad \dots \quad (2.1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0 \qquad \dots \quad (2.2)$$

$$\rho\left(\frac{\partial E}{\partial t}+u\frac{\partial E}{\partial r}\right)-\frac{p}{\rho}\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}(rF)=0 \qquad \dots \quad (2.3)$$

where $p$, $\rho$, $u$ and $E$ denote the pressure, density, material velocity and internal energy per unit mass, respectively, and $F$ is the flux of radiation

For an ideal gas we have

$$\left.\begin{aligned} E &= \frac{p}{(\gamma-1)\rho} \\ p &= \rho \bar{R} T \end{aligned}\right\} \qquad \dots \quad (2.4)$$

Also assuming local thermodynamic equilibrium and taking the radiative diffusion approximation we have,

$$F=-\frac{4\lambda}{3}\frac{\partial}{\partial r}(\sigma T^4) \qquad \dots \quad (2.5)$$

where $\sigma$ is the Stefan-Boltzmann constant and $\lambda$ the mean free path of radiation.

## Similarity Transformations

We now introduce the similarity assumptions for this flow as follows

$$u=R^{\alpha}\Phi_1, \qquad \rho=\rho_0\psi$$

$$p=\rho_0 R^{\beta} f_1, \qquad F=\rho_0 R^{\delta}\xi_1 \qquad \dots \quad (3.1)$$

where $R$ is the radius of the shock wave forming the outer edge of the disturbance, $\rho_0$ the density of the undisturbed atmosphere, $\Phi_1$, $\psi$, $f_1$ and $\xi_1$ are functions of $\eta$ where,

$$\eta = r/R$$

Substituting (3.1) in (2.1) we get

$$\frac{1}{\eta}[\Phi_1+(\psi/\psi')(\Phi'_1+\Phi_1/\eta)] = R^{-\alpha}\,\frac{dR}{dt} \tag{3.2}$$

The left hand side is a function of $\eta$ only whereas the right hand side a function of $t$ only. This can be satisfied if

$$R^{-\alpha}\,.\,\frac{dR}{dt} = A \text{ (constant)} \tag{3.3}$$

Substituting (3.2), (3.3) in (2.2) we get

$$\psi[A\alpha\Phi_1+(\Phi_1-A\eta)\Phi'_1]R^{2\alpha}+f'_1R^{\beta} = 0 \qquad \ldots \quad (3.4)$$

In order that the equation may be satisfied identically we must have $\beta = 2\alpha$

Finally equation (2.3) is transformed to

$$[2\alpha Af_1+(\Phi_1-A\eta)f'_1+\gamma f_1(\Phi'_1+\Phi_1/\eta)]R^{3\alpha}+(\gamma-1)\frac{R^{\delta}}{\eta}\,.\,\frac{d}{d\eta}(\eta\xi_1) = 0$$

we find that $\delta = 3\alpha$ so as to satisfy the above equation identically.

Equation (3.2), (3.4) and (3.5) may be reduced to non-dimensional form by substituting

$$\begin{aligned} \Phi_1 &= A\Phi \\ f_1 &= A^2 f \\ \xi_1 &= A^3\xi \end{aligned} \qquad \ldots \quad (3.6)$$

The resulting equations which contain only one parameter $\gamma$ are

$$(\Phi'+\Phi/\eta)+(\Phi-\eta)\psi'/\psi = 0 \qquad \ldots \quad (3.7)$$

$$(\Phi-\eta)\Phi'+\alpha\Phi = -f'/\psi \qquad \ldots \quad (3.8)$$

$$2\alpha f+(\Phi-\eta)f'+\gamma f(\Phi'+\Phi/\eta)+\frac{(\gamma-1)}{\eta}\,.\,\frac{d}{d\eta}(\eta\xi) = 0 \qquad \ldots \quad (3.9)$$

We shall now discuss the similarity form of the equation introduced by the flux of radiation (equation 2.5). The mean free path of radiation, in general, is a complicated function of temperature and density. However, to simplify the problem it is assumed that $\lambda$ can be written in a form of power law.

$$\lambda = \lambda_1 T^m (\rho/\rho_0)^n \qquad \ldots \quad (3.10)$$

where
$$T = \frac{p}{\bar{R}\rho} = \frac{A^2 R^{2\alpha}}{\bar{R}} \cdot (f/\psi) \qquad \ldots \quad (3.11)$$

Substituting (3.1), (3.6), (3.10) and (3 11) in (2.5) we get

$$R^{3\alpha}\xi = -\frac{16\sigma\lambda_1}{3\rho_0} A^{2m+5} \cdot (\bar{R})^{-(m+4)} (f)^{-(m+3)} \psi^{n-m-5} (f'\psi - f\psi') R^{2\alpha(m+4)-1} \qquad (3.12)$$

This can be satisfied if $3\alpha = 2\alpha(m+4)-1$, which gives

$$m = -\frac{5}{2} + \frac{1}{2\alpha} \qquad \ldots \quad (3.13)$$

We therefore have

$$\xi = -Kf^{-(m+3)}\psi^{n-m-5}(f'\psi - f\psi') \qquad \ldots \quad (3.14)$$

where

$$K = \frac{16\sigma\lambda}{3\rho_0} A^{2m+5} \quad (\bar{R})^{-(m+4)}$$

The constant $\alpha$ is determined from total energy consideration and $n$ is then determined by fitting the mean free path law (equation 3.10) to mean free path-temperature data.

## Total Energy Considerations

The values of $A$ and $\alpha$ are determined from the rate of change of total energy of the configuration. Suppose the disturbance lies between radii $hR$ and $R$ where $h$ is a constant and $0 \leqslant h < 1$. Then the total energy is given by

$$E_T = 2\pi \int_{hR}^{R} (E + \tfrac{1}{2}u^2)\rho r d\gamma$$

$$= 2\pi\rho_0 A^2 R^{2\alpha+2} \int_h^1 \psi \left\{ \frac{f}{(\gamma-1)\psi} + \tfrac{1}{2}\Phi^2 \right\} \eta d\eta \qquad \ldots \quad (4.1)$$

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$$E_T = \rho_0 A^2 B R^{2\alpha+2} \qquad \ldots \quad (4.2)$$

If $E_T$ remains constant with time, then we have $\alpha = -1$ and

$$A^2 = \frac{E_T}{B\rho_0} \qquad \ldots \quad (4.3)$$

In this case it is possible to get a first integral from equations (3.7) to (3.9)

$$\xi = (\eta - \Phi)\psi \left[ \frac{f}{(\gamma-1)\psi} + \tfrac{1}{2}\Phi^2 \right] - f\Phi + \frac{\text{constant}}{\eta} \qquad \ldots \quad (4.4)$$

## Shock Conditions

We are considering here a problem where the disturbance is bounded by a strong shock front. Three boundary conditions are provided by relations at the shock and the remaining one by the value of the flux at some fixed boundary. We then have a double boundary value problem.

The three conditions at the shock are given by the principles of conservation of mass, momentum and energy across the shock, namely,

$$\rho_s(U-u_s) = \rho_0 U = m_s \qquad \ldots \quad (5.1)$$

$$p_s - p_0 = m_s \,.\, u_s \qquad \ldots \quad (5.2)$$

$$E_s + \frac{p_s}{\rho_s} + \tfrac{1}{2}(U-u_s)^2 - \frac{F_s}{m_s} = E_0 + \frac{p_0}{\rho_0} + \tfrac{1}{2}U^2 - \frac{F_0}{m_s} \qquad \ldots \quad (5.3)$$

where suffixes $s$ and $o$ denote conditions behind and in front of the shock, respectively, and $U$ is the shock velocity. We consider a very strong shock wave and as such $F_0 = P_0 = 0$, we get

$$u_s = \frac{\mu-1}{\mu} U \qquad (5.4)$$

$$p_s = \frac{\mu-1}{\mu} \rho_0 U^2 \qquad (5.5)$$

where $$\mu = \rho_s/\rho_0 \qquad (5.6)$$

Also equation (5.3) simplifies to

$$F_s = \frac{\mu-1}{\mu^2}\left[\frac{\gamma+1}{\gamma-1} - \mu\right] m_s U^2 \qquad (5.7)$$

If we put terms in their similarity form we get

$$f_s = \Phi_s = \frac{\psi_s - 1}{\psi_s} \qquad (5.8)$$

and

$$\xi_s = (\psi_s - 1)\left[\frac{\gamma+1}{\gamma-1} - \psi_s\right] \Big/ 2\psi_s^{\,3} \qquad (5\ 9)$$

From (5.9) $\xi_s$ is determined in terms of $\psi_s$ If $\xi_s$ were known explicitly in terms of $f_s$ and $\psi_s$ then (5.8) and (5.9) would be sufficient to determine $\psi_s$ and all other quantities behind the shock, when it involves a derivative as in the case of (2.5) we require another boundary condition to determine $\psi_s$.

THE SOLUTION

Equation (3.7) and (3.8) for $\alpha = -1$ become

$$('\Phi + \Phi/\eta) + (\Phi - \eta)\psi'/\psi = 0 \qquad \dots \quad (6.1)$$

$$(\Phi - \eta)\Phi' - \Phi = -f'/\psi \qquad \dots \quad (6.2)$$

Instead of (3 9) we use the energy integral (4.4) which becomes

$$\xi = (\eta - \Phi)\psi\left[ \frac{f}{(\gamma-1)\psi} + \tfrac{1}{2}\Phi^2 \right] - f\Phi \qquad \dots \quad (6.3)$$

For $\alpha = -1$, equation (3 13) gives $m = -3$, the mean free path is then given by

$$\lambda = \lambda_1 T^{-3} \left( \frac{\rho}{\rho_0} \right)^n \qquad \dots \quad (6.4)$$

Since we have introduced radiative diffusion in the problem the mean free path of radiation is restricted in the form given by equation (6.4). This restriction fortunately allows for a variation of mean free path which is not unrealistic for temperature in air under $10^5$°C Equation (6.4) was fitted to the well known data for variation of mean free path in air with temperature and we got for $n$ a value $= -3/2$ Equation (3 14) then gives

$$\xi = -K\psi^{-7/2}(f'\psi - f\psi') \qquad \dots \quad (6.5)$$

where,

$$K = \frac{16\sigma\lambda}{3\rho_0}(A\bar{R})^{-1}$$

Eliminating $f'$ and $\psi'$ with the help of (6.1) and (6.2), equation (6.5) gives

$$\Phi'\left[ (\Phi-\eta) - \frac{f}{\psi(\Phi-\eta)} \right] = \Phi\left[ 1 + \frac{f}{\eta\psi(\Phi-\eta)} \right] + \frac{\xi\psi^{3/2}}{K} \qquad \dots \quad (6.6)$$

If for any value of $\eta$, $f$, $\Phi$, $\psi$ and $\xi$ are known their values can be computed step by step for other values of $\eta$.

Two of the boundary conditions required in integrating these equations are the shock relations (5.8). The third shock condition (5.9) is automatically satisfied by $\xi$ which is given by the relation (6.3) or (6.5). Assuming no heat source at the centre the third boundary condition is $\xi = 0$ at $\eta = 0$.

We, therefore, have a double boundary value problem. By suitably scaling the variables we can reduce it to a single boundary problem. We assume that pressure and density are finite and non-zero at the axis of summary *i.e.* at $\eta = 0$, $f = f_0$ and $\psi = \psi_0$, and make the following transformations.

$$\bar{\Phi} = \epsilon_0 \Phi, \qquad \bar{\eta} = \epsilon_0 \eta, \qquad \bar{\psi} = \psi/\psi_0$$
$$\bar{f} = f/f_0, \qquad \bar{\xi} = \frac{\epsilon_0 \xi}{f_0} \qquad \dots \ (6.7)$$

where $$c^2_0 = \psi_0/f_0 \qquad \dots \ (6.8)$$

Making use of these transformations, equations (6.1), (6.2) (6.3) and (6.6) become

$$(\bar{\Phi}-\bar{\eta})\bar{\psi}'/\bar{\psi} = -(\bar{\Phi}'+\bar{\Phi}/\bar{\eta}) \qquad \dots \ (6.9)$$

$$(\bar{\Phi}-\bar{\eta})\bar{\Phi}'-\bar{\Phi} = -\bar{f}/\bar{\psi} \qquad \dots \ (6.10)$$

$$\bar{\xi} = (\bar{\eta}-\bar{\Phi})\bar{\psi}\left[\frac{\bar{f}}{(\gamma-1)\bar{\psi}}+\tfrac{1}{2}\bar{\Phi}^2\right] - \bar{f}\,\bar{\Phi} \qquad \dots \ (6.11)$$

$$\left[(\bar{\Phi}-\bar{\eta}) - \frac{\bar{f}}{\bar{\psi}\,(\bar{\Phi}-\bar{\eta})}\right]\bar{\Phi}' = \bar{\Phi}\left[1+\frac{\bar{f}}{\bar{\eta}\,\bar{\psi}(\bar{\Phi}-\bar{\eta})}\right] + \frac{\bar{\xi}\,\bar{\psi}^{3/2}}{\bar{K}} \qquad \dots \ (6.12)$$

where, $$\bar{K} = Kf_0^{-1}\psi_0^{-3/2}$$

The boundary conditions become

$$\left.\begin{array}{l} \bar{\Phi} = \bar{\xi} = 0 \\ \bar{\psi} = \bar{f} = 1 \end{array}\right] \quad \text{at } \bar{\eta} = 0 \qquad \dots \ (6.13)$$

To find at that value of $\bar{\eta}$ the bounding shock occurs we used the shock conditions (5.8). The shock will occur where

$$\bar{\eta} = \epsilon_0 \eta = \epsilon_0 \qquad \dots \ (6.14)$$

$$\bar{\Phi} = \epsilon_0 \Phi = \epsilon_0\left(\frac{\psi_s-1}{\psi_s}\right) \qquad \dots \ (6.15)$$

and $$\bar{f}/\bar{\psi} = \frac{\epsilon_0^2 f}{\psi} = \epsilon_0^2\left(\frac{\psi_s-1}{\psi_s}\right) \qquad \dots \ (6.16)$$

Hence the shock occurs where

$$\bar{f}/\bar{\psi} = \bar{\Phi}(\bar{\eta}-\bar{\Phi}) \qquad \dots \ (6.17)$$

and the values of the scaling factors and the density ratio across the shock can be obtained from (6.14), (6.15) and (6.8).

Analytical solution of the equations (6.9) to (6.12) near the axis of symmetry was obtained. It is of the form

$$\bar{\Phi} = a_0\bar{\eta}^3, \quad \bar{\Psi} = 1+2a_0\bar{\eta}^2$$

$$\bar{f} = 1+a_0\eta^4 \qquad \ldots \quad (6.18)$$

where

$$a_0 = \frac{1}{4\bar{K}(\gamma-1)}$$

The approximate solution near the centre of symmetry given by (6.18) shows that

$$\bar{\Phi}' = \bar{f}' = \bar{\Psi}' = 0 \quad \text{when} \quad \bar{\eta} = 0 \qquad \ldots \quad (6.19)$$

and substituting these values of the derivatives at the centre, integration can be carried out by numerical means to obtain the variation of $\bar{\Phi}$, $\bar{f}$, $\bar{\Psi}$ and $\bar{\xi}$ with increasing $\bar{\eta}$, for various values of $\bar{K}$.

## Discussion of Results

For the explosion in air, the value of $\gamma$ was taken as 1.2, this being approximately true for pressures of 10 to $10^4$ atmospheres and temperatures between 20,000 and 200,000°K. A series of values of $\bar{K}$, the constant introduced by the expression for radiation diffusion were used. The variation of $\Phi, f, \psi, \xi$ and $\bar{R}T$ with $\eta$ is given in figures 1 to 5. Dotted curves give Lin's solution for $\gamma = 1.2$.

For $\bar{K} = 500$ the effect of heat flux is substantial throughout the disturbance. The point of maximum heat flux is at the shock and the temperature is rendered almost uniform by transport of radiation energy. In this case the density ratio across the shock is 2.79, compared with 11 when there is no radiation at the shock so that the mass of air just inside the shock is less and an appreciable amount of engulfed material lies in the central part of the disturbance.

As $\bar{K}$ is decreased the point of maximum flux moves inwards from the shock and radiation becomes negiligible in the outer regions of the disturbance. The value of the density ratio across the shock also increases as $\bar{K}$ is decreased, for $\bar{K} = 100$ and 10 the density ratio being 3.85 and 10.83, respectively. For $\bar{K} = 10$, most of the engulfed air lies in a thin shell behind the shock and the pressure profile is approximately the same as that of no radiation flux case.

The particle velocity at the shock front decreases as $\bar{K}$ is increased. As one moves from the shock front towards the axis of symmetry, the particle velocity at first decreases very rapidly and then tends to zero as the axis of symmetry is approached. Moreover, for a higher value of $\bar{K}$, the particle velocity approaches the axis at points closer to the shock front.

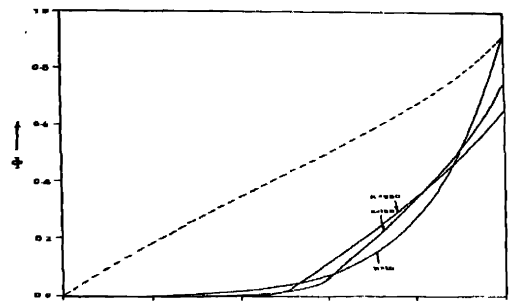

**Figure 1.** Variation of fluid velocity with radius.

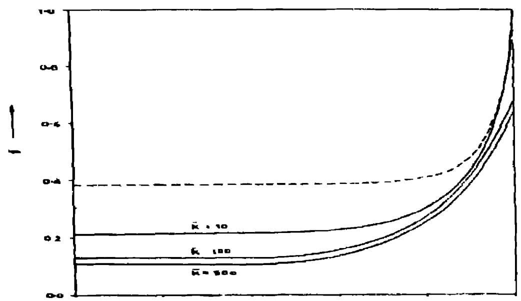

**Figure 2.** Variation of pressure with radius.

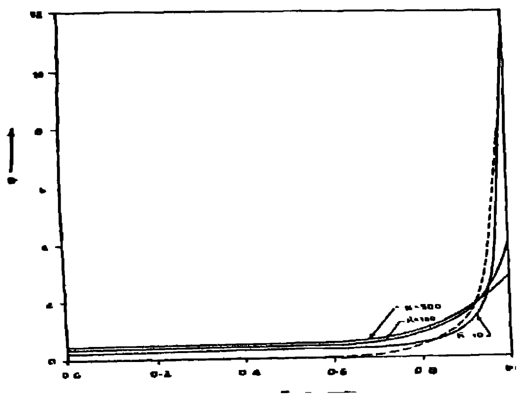

**Figure 3.** Variation of fluid density with radius.

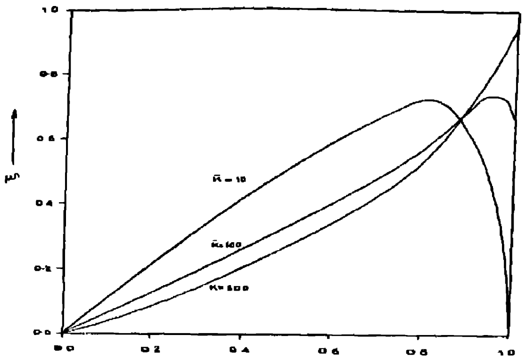


Figure 4. Variation of heat flux with radius.

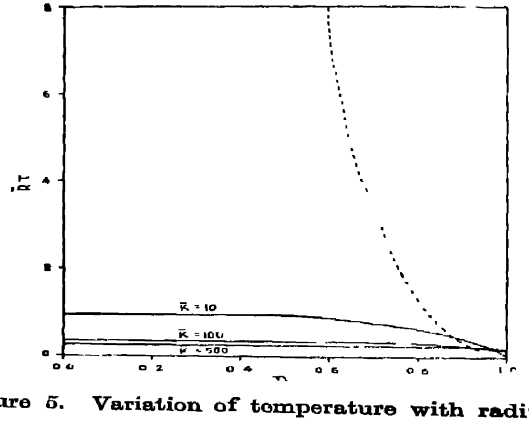


Figure 5. Variation of temperature with radius.

## Conclusions

When the effect of radiative heat flux is considered in the problem of an intense cylindrical explosion, it is found that the type of configuration obtained as well as the rate of propagation of the bounding shock depends on the amount of energy released and on the density of the ambient air. For a high energy explosion the solution is considerably affected by the radiative heat flux.

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